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THE D_1-TRIANGULATION OF $\mathbb{R}^n$ FOR SIMPLICIAL ALGORITHMS FOR COMPUTING SOLUTIONS OF NONLINEAR EQUATIONS*

CHUANGYIN DANG

We present a new triangulation of $\mathbf{R}^n$, which is called the D_1-triangulation, for computing zero points or fixed points of nonlinear mappings. The D_1-triangulation subdivides the unit cube and is based on very elementary pivot rules. We compare the D_1-triangulation to several well-known triangulations of $\mathbf{R}^n$ which triangulate the unit cube. According to several measures of efficiency the new triangulation is superior, such as the number of simplices in the unit cube, the diameter of a triangulation, the average directional density, and the surface density.

1. Introduction. There are now a number of simplicial algorithms for computing zero points or fixed points using triangulations of $\mathbb{R}^n$, for example, Merrill's homotopy restart method [5] and van der Laan and Talman's variable dimension simplicial restart algorithms without an extra dimension [4]. The other variable dimension algorithms have been introduced by Wright [9] and by Kojima and Yamamoto [3]. Allgower and Georg's paper [1] is an excellent survey of this field.

It has been accepted by now that the efficiency of the various simplicial homotopy and restart algorithms for solving equations is influenced in a critical manner by the triangulation employed. To evaluate and design triangulations for these algorithms, Todd, and Saigal, Solow and Wolsey established several measures in [6] and [7], such as the number of simplices in the unit cube, the diameter of a triangulation, the average directional density, and the surface density. Eaves and Yorke [2] showed that the average directional density and the surface density are equivalent.

To improve the efficiency of simplicial fixed point algorithms, we construct a new triangulation of $\mathbb{R}^n$ and show that according to these measures it is the best of the well-known triangulations of $\mathbb{R}^n$, which subdivide the unit cube.

In §2 the D_1-triangulation is introduced. We describe the pivot rules of the D_1-triangulation in §3. The number of simplices in the unit cube, the diameter, and the surface density are calculated in §§4, 5, and 6, respectively.

2. The D_1-triangulation of $\mathbb{R}^n$. Let $y^0, y^1, \ldots, y^k$ be a set of vectors in $\mathbb{R}^n$. If they are affinely independent, then we call their convex hull, σ, a k-simplex and write

$$\sigma = [y^0, y^1, \ldots, y^k] = \operatorname{conv}\{y^0, y^1, \ldots, y^k\}.$$

A simplex τ is called a face of a simplex σ if all vertices of τ are vertices of σ. If $\dim \tau = \dim \sigma - 1$, we call τ a facet of σ. In addition, if y is the vertex of σ which is not a vertex of τ, τ is called the facet of σ opposite y.

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Let C be a convex subset of $\mathbb{R}^n$ and let $\dim C = m$. We call G a triangulation of C if

(1) G is a collection of m-simplices,

(2) $C = \bigcup_{\sigma \in G} \sigma$,

(3) for any $\sigma^1, \sigma^2 \in G$, $\sigma^1 \cap \sigma^2$ is either empty or a common face of both σ^1 and σ^2,

(4) each $x \in C$ has a neighborhood meeting only a finite number of simplices of G.

We denote the collection of j-simplices that are faces of simplices of G by G^j, for $j = 0, 1, \ldots, m$.

For ease of notation, let $N = \{1, 2, \ldots, n\}$, let $D_1^{0c} = \{y \in \mathbb{R}^n \mid$ all components of y are even$\}$, and for $i = 1, 2, \ldots, n$, let u^i be the ith unit vector in $\mathbb{R}^n$.

As follows, we construct the simplices of a new triangulation of $\mathbb{R}^n$. We assume $n \geqslant 2$.

DEFINITION 2.1. Let s denote a sign vector in $\mathbb{R}^n$ such that $s_i \in \{-1, +1\}$ for all $i \in N$. Let $0 \leqslant p \leqslant n - 1$ be an integer. Let $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ be a permutation of the n elements of N such that $\pi(p) < \cdots < \pi(n)$ if $p \geqslant 1$ and $\pi(1) < \cdots < \pi(n)$ if $p = 0$.

Let $y \in D_1^{0c}$. If $p = 0$, let $y^0 = y$ and

$$y^k = y + s_{\pi(k)} u^{\pi(k)}, \qquad k = 1, 2, \ldots, n.$$

If $p \geqslant 1$, let

$$y^0 = y + s,$$

$$y^k = y^{k-1} - s_{\pi(k)} u^{\pi(k)}, \qquad k = 1, 2, \ldots, p - 1, \quad \text{and}$$

$$y^k = y + s_{\pi(k)} u^{\pi(k)}, \qquad k = p, \ldots, n.$$

LEMMA 2.1. *Let $y^0, y^1, \ldots, y^n$ be obtained from Definition 2.1. Then $y^0, y^1, \ldots, y^n$ are affinely independent.*

PROOF. If $p = 0$, then let

$$z^1 = y^1 - y^0 = s_{\pi(1)} u^{\pi(1)},$$

$$z^2 = y^2 - y^1 = s_{\pi(2)} u^{\pi(2)} - s_{\pi(1)} u^{\pi(1)},$$

$$\cdots$$

$$z^n = y^n - y^{n-1} = s_{\pi(n)} u^{\pi(n)} - s_{\pi(n-1)} u^{\pi(n-1)}.$$

Obviously, $z^1, \ldots, z^n$ are linearly independent.

If $p \geqslant 1$, then let

$$z^k = y^k - y^{k-1} = -s_{\pi(k)} u^{\pi(k)}, \qquad k = 1, 2, \ldots, p - 1,$$

$$z^p = y^p - y^{p-1} = -\sum_{k=p+1}^{n} s_{\pi(k)} u^{\pi(k)}, \quad \text{and}$$

$$z^k = y^k - y^{k-1} = s_{\pi(k)} u^{\pi(k)} - s_{\pi(k-1)} u^{\pi(k-1)}, \qquad k = p + 1, \ldots, n.$$

Suppose that $z^1, \ldots, z^n$ are linearly dependent. Then there exists a $q = (q_1, \ldots, q_n)^T \neq 0$ such that $q_1 z^1 + \cdots + q_n z^n = 0$. If $p = n - 1$, it is necessary that $q_1 = \cdots = q_{n-2} = 0$, $-q_{n-1} + q_n = 0$ and $q_n = 0$. We conclude that $q_1 = \cdots = q_n = 0$. If $p < n - 1$, we must have that $q_1 = \cdots = q_{p-1} = 0$, $q_{p+1} = 0$, $q_k - q_{k+1} - q_p = 0$ for $k = p + 1, \ldots, n - 1$, and $q_n - q_p = 0$. Therefore, $q_p = q_n$, $q_{n-1} = 2q_n$, $q_{n-2} = 3q_n, \ldots, q_{p+2} = (n - (p + 1))q_n$, and $q_{p+2} + q_p = 0$. Hence, $(n - (p + 1) + 1)q_n = 0$. Since $p < n - 1$, we have $q_1 = q_2 = \cdots = q_n = 0$. Thus the hypothesis is incorrect, i.e., $z^1, \ldots, z^n$ are linearly independent. Therefore, $y^0, y^1, \ldots, y^n$ are affinely independent. The proof is completed. □

Let $y^0, y^1, \ldots, y^n$ be obtained from Definition 2.1. Then their convex hull is an n-simplex by Lemma 2.1, which is denoted by $D_1(y, \pi, s, p)$. Let D_1 be the collection of all such simplices $D_1(y, \pi, s, p)$.

LEMMA 2.2. $\bigcup_{\sigma \in D_1} \sigma = \mathbb{R}^n$.

PROOF. Let x be an arbitrary point of $\mathbb{R}^n$. For each $i \in N$, let

$$y_i = \begin{cases} \lfloor x_i \rfloor & \text{if } \lfloor x_i \rfloor \text{ is even,} \\ \lfloor x_i \rfloor + 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad s_i = \begin{cases} +1 & \text{if } \lfloor x_i \rfloor \text{ is even,} \\ -1, & \text{otherwise.} \end{cases}$$

We have $0 \leqslant \operatorname{diag}(s_1, \ldots, s_n)(x - y) \leqslant u$, where $u = (1, \ldots, 1)^T$. Let π' be a permutation of N such that

$$0 \leqslant s_{\pi'(1)}(x_{\pi'(1)} - y_{\pi'(1)}) \leqslant \cdots \leqslant s_{\pi'(n)}(x_{\pi'(n)} - y_{\pi'(n)}) \leqslant 1.$$

If $\sum_{i=1}^n s_i(x_i - y_i) \leqslant 1$, let

$$q_1' = s_{\pi'(1)}(x_{\pi'(1)} - y_{\pi'(1)}), \ldots, q_n' = s_{\pi'(n)}(x_{\pi'(n)} - y_{\pi'(n)}),$$

and $q_0' = 1 - \sum_{j=1}^n q_j'$. Obviously, $q_j' \geqslant 0$ for all j and $\sum_{j=0}^n q_j' = 1$. Let $\pi = (1, 2, \ldots, n)$, $p = 0$, $y^0 = y$, and $y^k = y + s_k u^k$ for $k = 1, 2, \ldots, n$. It is easily seen that $x = \sum_{j=0}^n q_j y^j$, where $q_0 = q_0'$ and, for $j = 1, \ldots, n$, $q_j = q_h'$ with h the index for which $\pi'(h) = j$. Thus $x \in D_1(y, \pi, s, p)$.

If $\sum_{i=1}^n s_i(x_i - y_i) \geqslant 1$, then we show that there exists an integer $1 \leqslant p \leqslant n - 1$ such that the following system has a nonnegative solution:

$$\sum_{i=0}^{j-1} q_i' = s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}), \qquad j = 1, \ldots, p - 1,$$

$$\sum_{i=0}^{p-1} q_i' + q_k' = s_{\pi'(k)}(x_{\pi'(k)} - y_{\pi'(k)}), \qquad k = p, \ldots, n,$$

$$q_0' + q_1' + \cdots + q_n' = 1.$$

In fact, rewriting the system, we obtain

$$q_0' = s_{\pi'(1)}(x_{\pi'(1)} - y_{\pi'(1)}),$$

$$q_{j-1}' = s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) - s_{\pi'(j-1)}(x_{\pi'(j-1)} - y_{\pi'(j-1)}), \qquad j=2,\dots,p-1,$$

$$q_{p-1}' = -s_{\pi'(p-1)}(x_{\pi'(p-1)} - y_{\pi'(p-1)}) + \left(\sum_{j=p}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) - 1\right)\Big/(n-p),$$

$$q_k' = s_{\pi'(k)}(x_{\pi'(k)} - y_{\pi'(k)}) + \left(1 - \sum_{j=p}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)})\right)\Big/(n-p), \qquad k=p,\dots,n.$$

Let $N_0 = \{0,1,\dots,n\}$. If $q_{n-2}' \geqslant 0$ for $p = n-1$, it is clear that $q_j' \geqslant 0$ for all $j \in N_0$; otherwise, there exists a p_0, $1 \leqslant p_0 \leqslant n-2$, such that

$$-s_{\pi'(p_0-1)}(x_{\pi'(p_0-1)} - y_{\pi'(p_0-1)}) + \left(\sum_{j=p_0}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) - 1\right)\Big/(n-p_0) \geqslant 0$$

and

$$-s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) + \left(\sum_{j=p_0+1}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)}) - 1\right)\Big/(n-p_0-1) < 0.$$

Hence,

$$s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) + \left(1 - \sum_{j=p_0}^{n} s_{\pi'(j)}(x_{\pi'(j)} - y_{\pi'(j)})\right)\Big/(n-p_0)$$

$$\geqslant s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) + \big(1 - s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) - (n-p_0-1)s_{\pi'(p_0)}(x_{\pi'(p_0)} - y_{\pi'(p_0)}) - 1\big)/(n-p_0) = 0.$$

Therefore, by taking p equal to p_0, $q_j' \geqslant 0$ for all $j \in N_0$.

Let $1 \leqslant p \leqslant n-1$ be such that the system above has a nonnegative solution and let π be such that $\pi(k) = \pi'(k)$, $k = 1,2,\dots,p-1$, and $\pi(p) < \cdots < \pi(n)$.

Let

$$y^0 = y + s,$$

$$y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)}, \qquad k = 1,\dots,p-1,$$

$$y^k = y + s_{\pi(k)}u^{\pi(k)}, \qquad k = p,\dots,n.$$

Let q'_j be obtained from the system, for $j = 0, 1, \ldots, n$. Then it is easily seen that $x = \sum_{j=0}^{n} q_j y^j$, where $q_0 = q'_0$ and, for $j = 1, \ldots, n$, $q_j = q'_h$ with h the index for which $\pi'(h) = \pi(j)$. Thus $x \in D_1(y, \pi, s, p)$.

From these results, the lemma follows immediately. □

LEMMA 2.3. *For any σ^1 and $\sigma^2 \in D_1$, $\sigma^1 \cap \sigma^2$ is either empty or a common face of both σ^1 and σ^2.*

PROOF. Let $x \in \mathbb{R}^n$ be arbitrary. By Lemma 2.2, we may assume that $x \in \sigma$ for some

$$\sigma = [y^0, y^1, \ldots, y^n] = D_1(y, \pi, s, p),$$

i.e., $x = \sum_{i=0}^{n} q_i y^i$, with $q_i \geqslant 0$ for all i and $\sum_{i=0}^{n} q_i = 1$. Then x lies in a face of σ whose vertices are y^j for $j \in J := \{j \in N_0 | q_j > 0\}$. We show below how each y^j, $j \in J$, can be generated from x independent of y, π, s, and p. Thus these vertices are found for any simplex of D_1 containing x.

For each $i \in N$, let

$$r_i = \begin{cases} \lfloor x_i \rfloor & \text{if } \lfloor x_i \rfloor \text{ is even}, \\ \lfloor x_i \rfloor + 1 & \text{if } \lfloor x_i \rfloor \text{ is odd}, \end{cases}$$

and

$$t_i = \begin{cases} +1 & \text{if } x_i - r_i > 0, \\ \ \ 0 & \text{if } x_i - r_i = 0, \\ -1 & \text{if } x_i - r_i < 0. \end{cases}$$

Let $w = \sum_{i=1}^{n} t_i(x_i - r_i)$. Further, let

$$y_i(t_j) = \begin{cases} r_i + t_j & \text{if } i = j, \\ r_i & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, n$, and let $y(t_j) = (y_1(t_j), \ldots, y_n(t_j))^T$. Then

$$\{y(t_1), \ldots, y(t_n), r\} = \{y^j | j \in J\} \quad \text{if } w < 1, \quad \text{and}$$

$$\{y(t_1), \ldots, y(t_n)\} \setminus \{r\} = \{y^j | j \in J\} \quad \text{if } w = 1.$$

Suppose that $w > 1$. Let $T_1, \ldots, T_g$ be subsets of N such that $\bigcup_{k=1}^{g} T_k = N$ and for each $1 \leqslant k \leqslant g$, $t_i(x_i - r_i) = t_j(x_j - r_j)$ if $i \in T_k$ and $j \in T_k$ and for any $1 \leqslant e < f \leqslant g$, $t_i(x_i - r_i) < t_j(x_j - r_j)$ if $i \in T_e$ and $j \in T_f$. Let $T_0 = \varnothing$. Let $i(k) \in T_k$ for $k = 0, \ldots, g$. Since $w > 1$, there exist unique $0 \leqslant v < g$ and $q \geqslant 0$ such that

$$\begin{aligned} &t_{i(v)}(x_{i(v)} - r_{i(v)}) + (1 - |T_{v+1}| - \cdots - |T_g|)q \\ &\quad + |T_{v+1}|(t_{i(v+1)}(x_{i(v+1)} - r_{i(v+1)}) - t_{i(v)}(x_{i(v)} - r_{i(v)})) \\ &\quad + \cdots + |T_g|(t_{i(g)}(x_{i(g)} - r_{i(g)}) - t_{i(v)}(x_{i(v)} - r_{i(v)})) = 1, \end{aligned}$$

and

$$t_{i(j)}(x_{i(j)} - r_{i(j)}) - t_{i(v)}(x_{i(v)} - r_{i(v)}) - q \geqslant 0, \qquad j = v + 1, \ldots, g.$$

For $0 \leqslant k \leqslant v$, let for $i = 1, \ldots, n$,

$$y_i(T_k) = \begin{cases} r_i + t_i & \text{if } i \notin T_0 \cup T_1 \cup \cdots \cup T_k, \\ r_i & \text{otherwise,} \end{cases}$$

and let $y(T_k) = (y_1(T_k), \ldots, y_n(T_k))^T$. For $v + 1 \leqslant k \leqslant g$, let for each $j \in T_k$,

$$\hat{y}_i(j) = \begin{cases} r_i + t_j & \text{if } i = j, \\ r_i & \text{otherwise,} \end{cases}$$

and for all i, let $\hat{y}(j) = (\hat{y}_1(j), \ldots, \hat{y}_n(j))^T$. Let

$$\bar{g} = \begin{cases} g - 1 & \text{if } t_{i(j)}(x_{i(j)} - r_{i(j)}) - t_{i(v)}(x_{i(v)} - r_{i(v)}) - q = 0 \quad \text{for } j = g, \\ g & \text{otherwise.} \end{cases}$$

If $q = 0$, then

$$\{y(T_k) | 0 \leqslant k < v\} \cup \left(\bigcup_{k=v+1}^{\bar{g}} \{\hat{y}(j) | j \in T_k\} \right) = \{y^j | j \in J\},$$

and if $q > 0$, then

$$\{y(T_k) | 0 \leqslant k \leqslant v\} \cup \left(\bigcup_{k=v+1}^{\bar{g}} \{\hat{y}(j) | j \in T_k\} \right) = \{y^j | j \in J\}.$$

From these results, we obtain the proof of the lemma. □

THEOREM 2.4. *D_1 is a triangulation of $\mathbb{R}^n$.*

PROOF. Let $x \in \mathbb{R}^n$ be arbitrary. It is clear that x is only contained in a finite number of simplices of D_1. Using Lemma 2.1, Lemma 2.2, and Lemma 2.3, we complete the proof of the theorem. □

The D_1-triangulation of $\mathbb{R}^3$ is illustrated in Figure 1.

3. The pivot rules of the D_1-triangulation. Let $\sigma = [y^0, y^1, \ldots, y^n] = D_1(y, \pi, s, p)$ be given. We wish to obtain the unique n-simplex

$$\bar{\sigma} = [\bar{y}^0, \bar{y}^1, \ldots, \bar{y}^n] = D_1(\bar{y}, \bar{\pi}, \bar{s}, \bar{p}),$$

containing all vertices of σ except y^i. Table 1 shows how $\bar{y}$, $\bar{\pi}$, $\bar{s}$, and $\bar{p}$ depend on y, π, s, p, and i. From this table it is easy to obtain each vertex $\bar{y}^k$, $k = 0, 1, \ldots, n$, of $\bar{\sigma}$, and in particular its new vertex.

4. Comparison of the numbers of simplices in the unit cube. Let $I^n = \{x \in \mathbb{R}^n | 0 \leqslant x \leqslant u\}$ be the unit cube in $\mathbb{R}^n$.

THEOREM 4.1. *The number of simplices of the D_1-triangulation in the unit cube is equal to*

$$d_n = n + n(n-1) + \cdots + n(n-1) \cdots 4 \cdot 3 + 2.$$

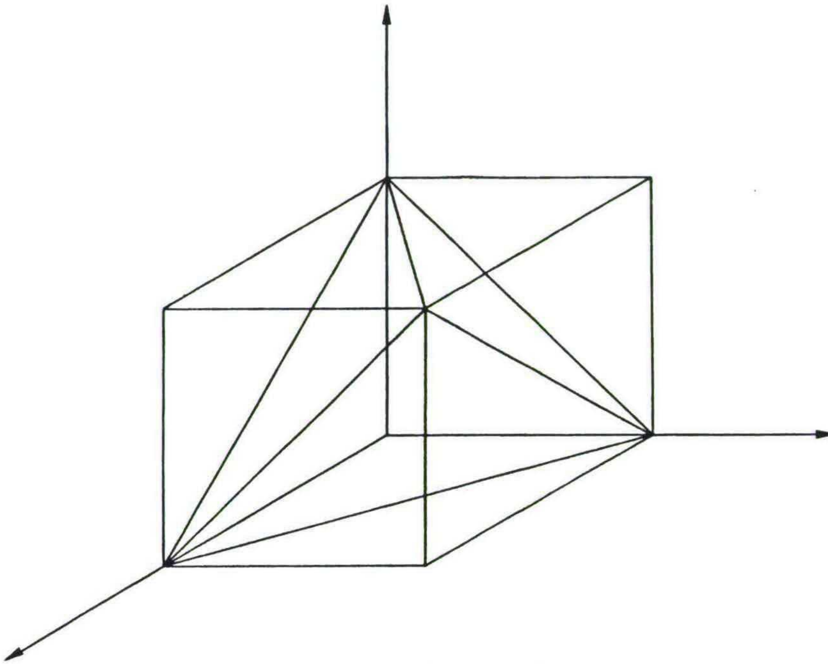

FIGURE 1. D_1-Triangulation of the Unit Cube in $\mathbf{R}^3$.

TABLE 1

The Pivot Rules of the D_1-Triangulation

p	i	$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$
0	0	y	s	π	$p+1$
0	$i \geq 1$	y	$s - 2s_{\pi(i)}u^{\pi(i)}$	π	p
1	0	y	s	π	$p-1$
$2 \leq p \leq n-1$	0	y	$s - 2s_{\pi(1)}u^{\pi(1)}$	π	p
$2 \leq p \leq n-1$	$1 \leq i < p-1$	y	s	$(\pi(1), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$	p
$2 \leq p \leq n-1$	$p-1$	y	s	$(\pi(1), \ldots, \pi(p-2), \pi(p), \ldots, \pi(j), \pi(p-1), \pi(j+1), \ldots, \pi(n))$*	$p-1$
$1 \leq p < n-1$	$i > p-1$	y	s	$(\pi(1), \ldots, \pi(p-1), \pi(i), \pi(p), \ldots, \pi(i-1), \pi(i+1), \ldots, \pi(n))$	$p+1$
$n-1$	$n-1$	$y + 2s_{\pi(n)}u^{\pi(n)}$	$s - 2s_{\pi(n)}u^{\pi(n)}$	π	p
$n-1$	n	$y + 2s_{\pi(n-1)}u^{\pi(n-1)}$	$s - 2s_{\pi(n-1)}u^{\pi(n-1)}$	π	p

*Where j is such that $\pi(j) < \pi(p-1) < \pi(j+1)$.

PROOF. Let $Q = \{D_1(y, \pi, s, p) | y = 0,\ s = (1, 1, \ldots, 1)^T\}$.

From Definition 2.1, in Q, there is only one simplex for which $p = 0$, one simplex for which $p = 1$, and $n!/(n - q + 1)!$ simplices for which $p = q$, $2 \leqslant q \leqslant n - 1$. Thus

$$|Q| = 1 + 1 + n!/(n - 1)! + n!/(n - 2)! + \cdots + n!/2!$$

$$= 2 + n + n(n - 1) + \cdots + n(n - 1) \cdots 4 \cdot 3.$$

Since $\bigcup_{\sigma \in Q} \sigma = I^n$, the proof of the theorem follows immediately. □

For the definitions of the K_1-, J_1- and H_1-triangulations, we refer to [8].

THEOREM 4.2. *The number of simplices in I^n of Freudenthal's K_1-triangulation, that of Tucker's J_1-triangulation, and that of Saigal's H_1-triangulation is $n!$.*

THEOREM 4.3. *If $n \geqslant 3$, then $d_n < n!$. As n goes to infinity, $d_n/n!$ converges to $e - 2$.*

PROOF. For $n = 3$, we have $d_3 < 3!$, since $d_3 = 5$ and $3! = 6$. Suppose $d_{n-1} < (n - 1)!$. Thus $nd_{n-1} < n!$. From

$$nd_{n-1} = n(n - 1) + n(n - 1)(n - 2) + \cdots + n(n - 1) \cdots 4 \cdot 3 + 2n$$

$$= d_n + (n - 2),$$

we obtain $d_n < n!$, since $n \geqslant 3$. By the induction principle, the conclusion $d_n < n!$ for $n \geqslant 3$ follows directly. Furthermore,

$$d_n/n! = 1/(n - 1)! + 1/(n - 2)! + \cdots + 1/2! + 2/n!,$$

so $d_n/n!$ converges to $e - 2$ as n goes to infinity. □

From these results, we obtain that the number of simplices of the D_1-triangulation is the smallest for these triangulations.

5. The diameter of the D_1-triangulation. Let G be a triangulation of $\mathbb{R}^n$ such that its restriction to I^n, $G|I^n = \{\sigma \subset I^n | \sigma \in G\}$, triangulates I^n and all vertices of $G|I^n$ are vertices of I^n. Let τ and τ' be two facets of G in the boundary of I^n, ∂I^n. Let $\sigma_0, \sigma_1, \ldots, \sigma_m$ be a sequence of simplices of G such that σ_i and σ_{i-1} are adjacent, for $i = 1, 2, \ldots, m$. If τ is a facet of σ_0 and τ' a facet of σ_m, then we say that the sequence of $\sigma_0, \sigma_1, \ldots, \sigma_m$ is a path of length $m + 1$ from τ to τ'. We define the distance between τ and τ' to be the minimum length of a path between τ and τ'. The diameter of G is the maximal distance between any two facets in the boundary. It is denoted by diam(G).

THEOREM 5.1.

$$\mathrm{diam}(K_1) = 1 + n(n - 1)/2 = O(n^2),$$

$$\mathrm{diam}(J_1) = \mathrm{diam}(K_1),$$

$$\mathrm{diam}(H_1) \geqslant (n^3 - n + 6)/6 = O(n^3), \quad \textit{and}$$

$$\mathrm{diam}(D_1) = 2n - 3 = O(n).$$

PROOF. Let $\sigma = [y^0, y^1, \dots, y^n] = K_1(0, \pi)$ and $\tau = [y^0, \dots, y^{n-1}]$, where $\pi = (1, 2, \dots, n)$. Let

$$\bar{\sigma} = [\bar{y}^0, \bar{y}^1, \dots, \bar{y}^n] = K_1(0, \bar{\pi}) \quad \text{and} \quad \bar{\tau} = [\bar{y}^0, \dots, \bar{y}^{n-1}],$$

where $\bar{\pi} = (n, n-1, \dots, 1)$. Let $\sigma_1, \dots, \sigma_{m-1}$ in $G|I^n$ be such that σ_{i-1} and σ_i are adjacent for $i = 2, \dots, m-1$, σ and σ_1 are adjacent, and also σ_{m-1} and $\bar{\sigma}$. It is easily seen that the smallest m is equal to $n(n-1)/2$. The distance between τ and $\bar{\tau}$ is obviously the greatest of all distances between two facets in ∂I^n. Therefore, $\operatorname{diam}(K_1) = n(n-1)/2 + 1$.

Since $J_1|I^n$ is the same as $K_1|I^n$, $\operatorname{diam}(J_1) = \operatorname{diam}(K_1)$.

Let $\sigma = [y^0, y^1, \dots, y^n] = H_1(y^{2x}, \pi)$ and $\tau = [y^1, \dots, y^n]$, where $y^{2x} = (1, 0, \dots, 0)^T$ and $\pi = (1, 2, \dots, n)$. Let

$$\bar{\sigma} = [\bar{y}^0, \bar{y}^1, \dots, \bar{y}^n] = H_1(\bar{y}^{2x}, \bar{\pi}) \quad \text{and} \quad \bar{\tau} = [\bar{y}^0, \dots, \bar{y}^{n-1}],$$

where $\bar{y}^{2x} = (1, \dots, 1)^T$ and $\pi = (n, n-1, \dots, 1)$.

Let $\sigma_1, \dots, \sigma_{m-1}$ be a sequence such that σ_{i-1} and σ_i are adjacent for $i = 2, \dots, m-1$, σ and σ_1 are adjacent, and also σ_{m-1} and $\bar{\sigma}$. Then the smallest m is equal to $(n^3 - n + 6)/6 - 1$. Thus the distance between τ and $\bar{\tau}$ is $(n^3 - n + 6)/6$. This means $\operatorname{diam}(H_1) \geqslant O(n^3)$.

Finally, let $\sigma = [y^0, y^1, \dots, y^n] = D_1(y, \pi, s, p)$ and $\tau = [y^1, y^2, \dots, y^n]$, where $y = 0$, $s = (1, \dots, 1)^T$, $p = n - 1$, and $\pi = (1, 2, \dots, n)$. Let

$$\bar{\sigma} = [\bar{y}^0, \bar{y}^1, \dots, \bar{y}^n] = D_1(\bar{y}, \bar{\pi}, \bar{s}, \bar{p}) \quad \text{and} \quad \bar{\tau} = [\bar{y}^1, \dots, \bar{y}^n],$$

where $\bar{y} = 0$, $\bar{s} = (1, 1, \dots, 1)^T$, $\bar{p} = n - 1$, and $\bar{\pi} = (n, n-1, \dots, 3, 1, 2)$. Let $\sigma_1, \dots, \sigma_{m-1}$ be a sequence such that σ and σ_1, σ_{i-1} and σ_i for $i = 2, \dots, m-1$, and $\bar{\sigma}$ and σ_{m-1} are adjacent. Then the smallest m is equal to $2n - 4$. The distance between τ and $\bar{\tau}$ is obviously the greatest of all distances between two facets in ∂I^n. Therefore, $\operatorname{diam}(D_1) = 2n - 3$.

From these results, the theorem follows immediately. □

6. The average directional density and surface density. From Eaves and Yorke [2], we know that for a triangulation the average directional density and surface density are equivalent. We calculate below the surface density and obtain the average directional density from the surface density.

First we calculate the surface density of the D_1-triangulation. Let

$$\sigma^0 = [0, u^1, \dots, u^n], \qquad \sigma^1 = [u, u^1, \dots, u^n],$$

$$\sigma^2 = [u, u - u^1, u^2, \dots, u^n], \dots, \sigma^{n-1}$$

$$= [u, u - u^1, \dots, u - u^1 - u^2 - \cdots - u^{n-2}, u^{n-1}, u^n].$$

The volume of a simplex σ is denoted by $V(\sigma)$. The surface area of a simplex σ is denoted by $\mathrm{SA}(\sigma)$. Let

$$\tau_0^0 = [u^1, u^2, \dots, u^n], \qquad \tau_1^0 = [0, u^2, \dots, u^n], \dots, \tau_{n-1}^0 = [0, u^1, \dots, u^{n-2}, u^n],$$

$$\tau_n^0 = [0, u^1, \dots, u^{n-1}]$$

be the facets of σ^0. Then

$$\mathrm{SA}(\sigma^0) = \sum_{i=0}^{n} V(\tau_i^0) = nV(\tau_n^0) + V(\tau_0^0).$$

Clearly,

$$V(\tau_n^0) = (1/(n-1)!)|\det[u^1, u^2, \ldots, u^n]| = 1/(n-1)!,$$

and

$$V(\tau_0^0) = (1/(n-1)!)\left|\det\left[u/\sqrt{n}, u^2 - u^1, \ldots, u^n - u^1\right]\right| = \sqrt{n}/(n-1)!,$$

so $\mathrm{SA}(\sigma^0) = (n + \sqrt{n})/(n-1)!$. Since $V(\sigma^0) = 1/n!$, we obtain that

$$\mathrm{SA}(\sigma^0)/V(\sigma^0) = n(n + \sqrt{n}).$$

For $k = 2, \ldots, n-1$, let

$$\tau_j^k = [u, u - u^1, \ldots, u - u^1 - \cdots - u^{k-1}, u^k, \ldots, u^{j-1}, u^{j+1}, \ldots, u^n],$$
$$j = k, \ldots, n,$$

$$\tau_0^k = [u - u^1, \ldots, u - u^1 - \cdots - u^{k-1}, u^k, \ldots, u^n], \quad \text{and}$$

$$\tau_j^k = [u, u - u^1, \ldots, u - u^1 - \cdots - u^{j-1}, u - u^1 - \cdots - u^{j+1}, \ldots,$$
$$u - u^1 - \cdots - u^{k-1}, u^k, \ldots, u^n], \qquad j = 1, 2, \ldots, k-1,$$

denote the facets of σ^k. Then

$$\mathrm{SA}(\sigma^k) = V(\tau_0^k) + \sum_{j=1}^{k-1} V(\tau_j^k) + (n-k+1)V(\tau_n^k).$$

Let

$$q_1 = q_2 = \cdots = q_{n-k} = \left((n-k+1)^2 - 3(n-k+1) + 3\right)^{-1/2} \quad \text{and}$$

$$q_{n-k+1} = -\left((n-k+1)^2 - 3(n-k+1) + 3\right)^{-1/2}(n-k-1).$$

Then

$$V(\tau_n^k) = (1/(n-1)!)\left|\det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & q_1 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdot & & \cdot & 1 & \cdots & 0 & q_{n-k} \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & q_{n-k+1} \end{bmatrix}\right|$$

$$= \left((n-k+1)^2 - 3(n-k+1) + 3\right)^{1/2}/(n-1)!.$$

Further

$$V(\tau_0^k) = (1/(n-1)!)\left|\det\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \end{bmatrix}\right|$$

$$= (n-k)/(n-1)!.$$

Now suppose $1 \leqslant j \leqslant k-1$. If $j < k-1$, let $q_j = 1/\sqrt{2}$, $q_{j+1} = -1/\sqrt{2}$, and $q_{j+2} = \cdots = q_n = 0$. If $j = k-1$, let

$$q_{k-1} = -(n-k)\left((n-k+1)^2 - (n-k+1) + 1\right)^{-1/2} \quad \text{and}$$

$$q_k = \cdots = q_n = \left((n-k+1)^2 - (n-k+1) + 1\right)^{-1/2}.$$

Then for all $j \in \{1, \ldots, k-1\}$,

$$V(\tau_j^k) = (1/(n-1)!)\left|\det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & q_j \\ \vdots & \vdots & & \vdots & 1 & \ddots & \vdots & \vdots & & \vdots & \vdots \\ & & & & \vdots & \ddots & 1 & & & & \\ \cdot & \cdot & & \cdot & 0 & \cdots & 1 & 1 & \cdots & 1 & q_{k-1} \\ \cdot & \cdot & & \cdot & 0 & \cdots & 0 & 0 & \cdots & 1 & q_k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & q_n \end{bmatrix}\right|$$

$$= \begin{cases} (n-k)\sqrt{2}/(n-1)! & \text{if } j \neq k-1, \\ \left((n-k+1)^2 - (n-k+1) + 1\right)^{1/2}/(n-1)! & \text{if } j = k-1. \end{cases}$$

Thus,

$$\begin{aligned} \mathrm{SA}(\sigma^k) = {} & (n-k)/(n-1)! + (n-k+1) \\ & \times\left((n-k+1)^2 - 3(n-k+1) + 3\right)^{1/2}/(n-1)! \\ & + (k-2)(n-k)\sqrt{2}/(n-1)! \\ & + \left((n-k+1)^2 - (n-k+1) + 1\right)^{1/2}/(n-1)!. \end{aligned}$$

Moreover,

$$V(\sigma^k) = (1/n!)\left|\det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix}\right|$$

$$= (n-k)/n!.$$

Hence,

$$\begin{aligned} \mathrm{SA}(\sigma^k)/V(\sigma^k) = n\Big(&n-k+(n-k+1)\big((n-k+1)^2-3(n-k+1)+3\big)^{1/2} \\ &+(k-2)(n-k)\sqrt{2}+\big((n-k+1)^2 \\ &-(n-k+1)+1\big)^{1/2}\Big)/(n-k). \end{aligned}$$

Let

$$\tau_0^1 = [u^1,\ldots,u^n], \qquad \tau_1^1 = [u,u^2,\ldots,u^n],$$

$$\tau_2^1 = [u,u^1,u^3,\ldots,u^n],\ldots,\tau_n^1 = [u,u^1,\ldots,u^{n-1}]$$

be the facets of σ^1. Then

$$\mathrm{SA}(\sigma^1) = nV(\tau_n^1) + V(\tau_0^1).$$

Let

$$q_1 = \cdots = q_{n-1} = (n^2-3n+3)^{-1/2} \quad\text{and}\quad q_n = -(n-2)(n^2-3n+3)^{-1/2}.$$

Then

$$V(\tau_n^1) = (1/(n-1)!)\left|\det\begin{bmatrix} 0 & 1 & \cdots & 1 & q_1 \\ 1 & 0 & \cdots & 1 & q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & q_{n-1} \\ 1 & 1 & \cdots & 1 & q_n \end{bmatrix}\right|$$

$$= (n^2-3n+3)^{1/2}/(n-1)!,$$

$V(\tau_0^1) = n^{1/2}/(n-1)!$, and $V(\sigma^1) = (n-1)/n!$. Moreover,

$$\mathrm{SA}(\sigma^1) = \big(n(n^2-3n+3)^{1/2} + n^{1/2}\big)/(n-1)!.$$

TABLE 2

Comparison of the K_1-, J_1-, and D_1-Triangulations

Triangulation	Number of Simplices in a Unit Cube	Diameter of a Triangulation	Average Directional Density
$K_1(J_1)$	$n!$	$O(n^2)$	$n(2+(n-1)\sqrt{2})g_n$
D_1	$n + n(n-1) + \cdots + n(n-1) \cdots 4 \cdot 3 + 2$	$O(n)$	$\mathrm{SD}(D_1)g_n$

Hence,

$$\mathrm{SA}(\sigma^1)/V(\sigma^1) = n\left(n(n^2 - 3n + 3)^{1/2} + n^{1/2}\right)/(n-1).$$

From the above results we obtain that the surface density of the D_1-triangulation equals

$$\mathrm{SD}(D_1) = \max\{\mathrm{SA}(\sigma^i)/V(\sigma^i) | i = 0, 1, \ldots, n-1\}.$$

Let

$$g_n = \Gamma(n/2)/((n-1)\Gamma(1/2)\Gamma((n-1)/2)).$$

From [2] we know that the average directional density of a triangulation is g_n times its surface density. Hence, the average directional density of the D_1-triangulation is equal to

$$\mathrm{ADD}(D_1) = \mathrm{SD}(D_1)g_n.$$

It is well known that both the average directional density of the K_1-triangulation and the one of the J_1-triangulation are equal to $n(2+(n-1)\sqrt{2})g_n$. It is obvious that we have that $\mathrm{ADD}(D_1) < \mathrm{ADD}(K_1) = \mathrm{ADD}(J_1)$, and that $\mathrm{ADD}(D_1)/\mathrm{ADD}(K_1)$ converges to 1 as n goes to infinity. Thus, the average directional density of the D_1-triangulation is smaller than the one of the K_1- or the J_1-triangulation. Table 2 summarizes the results above.

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